JOURNAL OF APPROXIMATION THEORY 82, 60-87 (1995)

Total Positivity, Finite Reflection Groups, and a Formula of Harish-Chandra*

KENNETH I. GROSS[†]

Department of Mathematics & Statistics, University of Vermont, Burlington, Vermont 05405

AND

DONALD ST. P. RICHARDS[‡]

Division of Statistics, University of Virginia, Charlottesville, Virginia 22903

Communicated by Paul Nevai

Received September 7, 1993; accepted in revised form May 16, 1994

DEDICATED TO RICHARD ASKEY, ON THE OCCASION OF HIS 60TH BIRTHDAY

Let W be a finite reflection (or Coxeter) group and $K: \mathbb{R}^2 \to \mathbb{R}$. We define the concept of total positivity for the function K with respect to the group W. For the case in which $W = \mathfrak{S}_n$, the group of permutations on n symbols, this notion reduces to the classical formulation of total positivity. We prove a basic composition formula for this generalization of total positivity, and in the case in which W is the Weyl group for a compact connected Lie group we apply an integral formula of Harish-Chandra (*Amer. J. Math.* **79** (1957), 87-120) to construct examples of totally positive functions. In particular, the function $K(x, y) = e^{xy}$, $(x, y) \in \mathbb{R}^2$, is totally positive with respect to any Weyl group W. As an application of these results, we derive an FKG-type correlation inequality in the case in which W is the Weyl group of SO(5). © 1995 Academic Press, Inc.

1. INTRODUCTION

This paper continues our work on the theory of total positivity and its connections with noncommutative harmonic analysis. In our earlier papers

* The results contained in this paper were announced in *Abstracts of Papers Presented to the Amer. Math. Soc.* **14** (1993), 495, and were presented in August 1993 at the International Joint Mathematics Meetings, Vancouver, B.C.

[†] Research supported by National Science Foundation Grant DMS-9123387.

[‡] Research supported by National Science Foundation Grant DMS-9101740.

0021-9045/95 $12.00

[7, 16], we related the classical notion of total positivity to analysis on the unitary group $U(n)$. In the theory developed and applied in those papers, integration over the unitary group became a powerful technique for proving total positivity. Yet, despite its obvious utility, the appearance of the unitary group in the study of total positivity seemed an inspired act of providence. In this paper we broaden the group theoretic context to include all nonabelian compact Lie groups, to extend the classical concept of total positivity, and to establish new positivity theorems. From the standpoint of our prior work, this expanded perspective introduces a framework relative to which the association of the unitary group to the classical concept of total positivity is completely natural.

By way of background, we briefly examine the main theme in [7, 16]. Let $\mathscr{D} \subseteq \mathbb{R}^2$ and r a positive integer. A function $K\colon \mathscr{D} \to \mathbb{R}$ is *totally positive of order* r [12] if the $n \times n$ determinant

$$\det(K(s_j, t_k)) = \begin{vmatrix} K(s_1, t_1) & K(s_1, t_2) & \cdots & K(s_1, t_n) \\ K(s_2, t_1) & K(s_2, t_2) & \cdots & K(s_2, t_n) \\ \vdots & \vdots & & \vdots \\ K(s_n, t_1) & K(s_n, t_2) & \cdots & K(s_n, t_n) \end{vmatrix} \tag{1.1}$$

is nonnegative for all $n = 1, \dots, r$ and for all $s_1 > \cdots > s_r$ and $t_1 > \cdots > t_r$ such that $(s_j, t_k) \in \mathscr{D}$. For the case in which $K(x, y) = f(xy)$ where $f\colon \mathbb{R} \to \mathbb{R}$ is a real-analytic function, we associated to the function f a sequence of real-valued functions ψ_n such that for all $n = 1, 2, \dots$

$$\det(K(s_j, t_k)) = V(s)\, V(t) \int_{U(n)} \psi_n(usu^{-1}t)\, du. \tag{1.2}$$

Here, du denotes Haar measure on $U(n)$ normalized to have total volume one; $s = \operatorname{diag}(s_1, \dots, s_n)$ and $t = \operatorname{diag}(t_1, \dots, t_n)$ are diagonal matrices such that all products $s_j t_k (j, k = 1, \dots, n)$ lie in the domain of f; and $V(s) = \prod_{1 \leq j < k \leq n} (s_j - s_k)$ denotes the Vandermonde determinant. It is then obvious that the function K is totally positive of order r if the functions ψ_n are nonnegative, $n = 1, \dots, r$. We then applied (1.2) in the case in which the function f is a classical hypergeometric function. For in that context the functions ψ_n can be expressed in terms of hypergeometric functions of matrix argument [6, 7] which, for appropriate values of the parameters and argument, can be shown to be positive. These results are noteworthy, not only in regard to the relevancy of the unitary group, but also in that the theory of hypergeometric functions of matrix argument leads to new results for classical generalized hypergeometric series.

We turn now to the present work. Let W be a finite reflection group; that is, a finite subgroup of the orthogonal group. In this paper we define the notion of total positivity with respect to W of a function $K\colon \mathbb{R}^2 \to \mathbb{R}$. To motivate the definition, let us observe that the symmetric group $\mathfrak{S}_n$ on n symbols is an example of a reflection group, that the generalized orthant $\{(s_1, ..., s_n): s_1 > \cdots > s_n\}$ is a fundamental Weyl chamber (cf. (2.2) below) for $\mathfrak{S}_n$, and that the determinant in (1.1) may be viewed as an alternating sum over $\mathfrak{S}_n$. We now make the following general definition: For any finite reflection group W, we will say that a function $K\colon \mathbb{R}^2 \to \mathbb{R}$ is *totally positive with respect to* W if an analogous alternating sum over W (given by (3.1) below) is nonnegative whenever the vectors $(s_1, ..., s_n)$ and $(t_1, ..., t_n)$ belong to a fundamental Weyl chamber of W. The connection between this viewpoint and the results of [7, 16] arises in the case in which W is the Weyl group of a compact Lie group. In this situation, we apply an integral formula of Harish-Chandra [8] to prove that the function $K(x, y) = e^{xy}$ on $\mathbb{R}^2$ is totally positive with respect to W. In the case in which W is the Weyl group of $\mathrm{U}(n)$ we recover the proof given in [7].

Let us briefly indicate how our results lead to new inequalities for totally positive functions. Consider again, for example, the case in which $K(x, y) = e^{xy}$, and let

$$\delta(s_1, s_2; t_1, t_2) = \begin{vmatrix} e^{s_1 t_1} & e^{s_1 t_2} \\ e^{s_2 t_1} & e^{s_2 t_2} \end{vmatrix}.$$

Since K is TP_2 in the classical sense, $\delta(s_1, s_2; t_1, t_2) > 0$ for $s_1 > s_2$, $t_1 > t_2$. Now define

$$\begin{aligned} D_K(s_1, s_2; t_1, t_2) = \delta(s_1, s_2; t_1, t_2) - \delta(-s_1, s_2; t_1, t_2) \\ - \delta(s_1, -s_2; t_1, t_2) + \delta(-s_1, -s_2; t_1, t_2). \end{aligned}$$

If $s_1 > s_2 > 0$ and $t_1 > t_2 > 0$ then, by ordinary total positivity, $\delta(s_1, s_2; t_1, t_2) > 0$, $\delta(-s_1, s_2; t_1, t_2) < 0$, $\delta(s_1, -s_2; t_1, t_2) > 0$ and $\delta(-s_1, -s_2; t_1, t_2) < 0$. However, we will deduce from Harish-Chandra's formula that $D_K(s_1, s_2; t_1, t_2) > 0$ for all $s_1 > s_2 > 0$ and $t_1 > t_2 > 0$, so that we have the stronger inequality

$$\begin{aligned} \delta(s_1, s_2; t_1, t_2) - \delta(-s_1, s_2; t_1, t_2) \\ > |\delta(s_1, -s_2; t_1, t_2) - \delta(-s_1, -s_2; t_1, t_2)|. \end{aligned} \tag{1.3}$$

Of course, (1.3) can be proved by elementary methods; indeed, it is equivalent to the fact that $\sinh x > 0$ for $x > 0$. However, by means of alternating sums over finite reflection groups, we will obtain more general inequalities, not only in the case of the function $K(x, y) = e^{xy}$ but also for broader classes of functions.

We close the introduction by describing the layout of the results to follow. In order to provide a self-contained treatment of our results, we present in Section 2 a brief exposé of the theory of abstract root systems, finite reflection groups, and Weyl chambers. In Section 3, we introduce the alternating sums which generalize the classical determinants, establish generalizations of two well-known properties of determinants and derive a generalization of the Binet–Cauchy formula. We give the definition of total positivity for finite reflection groups and we also derive a basic composition formula for the new definition of total positivity. Section 4 treats background material on Lie groups and Lie algebras, and Harish–Chandra's integral formula. In Section 5, we give a detailed statement of Harish–Chandra's formula for the groups of unitary and orthogonal matrices. In Section 6, we provide numerous examples of totally positive functions. For the unitary groups, we show that the resulting notion of total positivity coincides with the classical definition. In the case of the orthogonal groups, our definition of total positivity can be rewritten as the requirement that certain alternating sums of determinants are nonnegative.

One of the major applications of the classical theory of total positivity is to the derivation of correlation inequalities for probability distributions on $\mathbb{R}^n$ [2, 10, 13]. Therefore, it is natural to speculate that the new concepts of total positivity introduced here should also have similar probabilistic applications. In Section 7, we briefly recall the classical FKG correlation inequality on $\mathbb{R}^n$. Then we utilize total positivity with respect to the Weyl group W of $SO(5)$ to derive an analog for W of the FKG inequality on $\mathbb{R}^n$.

Finally, it is with great pleasure that, on the occasion of his sixtieth birthday, we dedicate this paper to Richard Askey. We do this not only because of his many kindnesses to us, but also because he has been a true leader in the art of proving positivity.

2. Finite Reflection Groups

We begin with some background material on abstract root systems. Proofs of our assertions, and the fully developed theory, can be found in most books on Lie groups and Lie algebras; cf. [3, 5, 11, 15].

Let E be an n-dimensional Euclidean space with inner product $(\cdot, \cdot)$. A *hyperplane* in E is a set of the form $P_\alpha = \{v \in E : (v, \alpha) = 0\}, \alpha \in E \setminus \{0\}$. A *reflection* in E is an invertible linear transformation σ_α, leaving pointwise fixed some hyperplane P_α and sending any vector orthogonal to P_α into its negative. Each reflection is an orthogonal transformation, given by the formula

$$\sigma_\alpha(v) = v - \frac{2(v, \alpha)}{(\alpha, \alpha)} \alpha$$

for all $v \in \mathrm{E}$.

2.1. DEFINITION. A subset Δ of the space E is a *(reduced) root system* if the following conditions hold:

(R1) Δ is finite, spans E, and $0 \notin \Delta$;

(R2) If $\alpha, \beta \in \Delta$ are proportional, then $\alpha = \beta$ or $\alpha = -\beta$;

(R3) If $\alpha \in \Delta$, the reflection σ_α leaves Δ invariant;

(R4) If $\alpha, \beta \in \Delta$, $2(\beta, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$.

The elements of Δ are called *roots*. We will always assume that Δ is *irreducible*; that is, Δ cannot be written as a nontrivial disjoint union, $\Delta = \Delta_1 \cup \Delta_2$, relative to which every root in Δ_1 is orthogonal to every root in Δ_2.

Let GL(E) denote the group of all invertible linear transformations of E. For any root system $\Delta \subset \mathrm{E}$, let W denote the subgroup of GL(E) generated by the set of reflections $\{\sigma_\alpha : \alpha \in \Delta\}$. It follows from property (R3) that W permutes the set Δ, so that W can be identified with a subgroup of the symmetric group on the finite set Δ. Hence W is a finite group, called the *Weyl* (or *Coxeter*) group of Δ.

A subset Ψ of Δ is called a *base of simple roots*, or a *base*, if Ψ is a vector space basis of E and each root β can be written as

$$\beta = \sum_{\alpha \in \Psi} k_\alpha \alpha \tag{2.1}$$

where the coefficients k_α are either all nonpositive or else all nonnegative integers. Every root system has a base. Moreover, the number of elements in a base, which equals the dimension of E, is referred to as the *rank* of Δ.

If all the coefficients k_α in (2.1) are nonnegative, we say that β is a *positive root*. The collection of positive roots is denoted by Δ_+. A more abstract approach to defining a system of positive roots is the following. Let $\{e_1, ..., e_n\}$ be a fixed basis of E. For $v \in \mathrm{E}$, we say that v is *positive*, and write $v > 0$, if $v = \sum_{j=1}^n x_j e_j$ with $x_1 = \cdots = x_k = 0$ and $x_{k+1} > 0$ for some $k \geqslant 0$. A root $\alpha \in \Delta$ is a *positive root* if it is positive as a vector in E.

The hyperplanes P_α, $\alpha \in \Delta$, partition E into a finite number of subsets. The connected components of $\mathrm{E} \setminus \bigcup_\alpha P_\alpha$ are called the *Weyl chambers* of E. Each Weyl chamber is an open convex subset of E. If C is a Weyl chamber then the *walls* of C are those subsets $\bar{C}\setminus\bigcap P_\alpha$ which have dimension $n-1$. Each vector $v \in \mathrm{E}\setminus\bigcup_\alpha P_\alpha$ belongs to a unique Weyl chamber, denoted by $\mathfrak{C}(v)$. The Weyl group W is the group of permutations of all Weyl chambers. Moreover, W acts *simply transitively* on the set of Weyl chambers; that is, for any pair of Weyl chambers C_1, C_2, there exists a unique $w \in W$ such that $wC_1 = C_2$.

For a given base Ψ, the chamber

$$\mathfrak{C}(\Psi) = \{v \in \mathrm{E}: (v, \alpha) > 0 \text{ for all } \alpha \in \Psi\} \tag{2.2}$$

is called the *fundamental Weyl chamber*.

From now on, we assume that a base Ψ has been chosen and we denote the corresponding fundamental Weyl chamber by $\mathfrak{C}$.

2.2. EXAMPLE. Consider the usual n-dimensional Euclidean space $\mathbb{R}^n$, and let $\{e_1, ..., e_n\}$ be the usual standard orthonormal basis of coordinate vectors. Denote by $x = (x_1, ..., x_n)$ the vectors in $\mathbb{R}^n$. Set $\Delta = \{e_j - e_k: 1 \leqslant j \neq k \leqslant n\}$. Then Δ is an abstract root system of rank $n-1$ for the subspace $\mathrm{E} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n: x_1 + \cdots + x_n = 0\}$. A system of positive roots is $\Delta_+ = \{e_i - e_j: 1 \leqslant i < j \leqslant n\}$ and a base is $\Psi = \{e_j - e_{j+1}: 1 \leqslant j \leqslant n-1\}$. A fundamental Weyl chamber is $\mathfrak{C} = \{(x_1, ..., x_n) \in \mathrm{E}: x_1 > x_2 > \cdots > x_n\}$. The Weyl group for Δ is $\mathfrak{S}_n$, the symmetric group on n symbols, so that each Weyl chamber is of the form $C = \{(x_1, ..., x_n) \in \mathrm{E}: x_{w(1)} > x_{w(2)} > \cdots > x_{w(n)}\}$, where $w \in \mathfrak{S}_n$.

In the general classification of irreducible root systems, or their associated finite reflection groups, this example is a root system of *type* A_{n-1}.

We close this section with some general remarks on root systems. In the general classification theory, the irreducible finite reflection groups are categorized as belonging to various *types*. These types include five infinite families: A_n, $n \geqslant 1$; B_n, $n \geqslant 2$; C_n, $n \geqslant 2$; D_n, $n \geqslant 4$; H_n^2, $n \geqslant 5$, $n \neq 6$; and seven additional types, denoted by E_6, E_7, E_8, F_4, G_2, I_3, and I_4. Of these, the root systems of type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, and G_2 are naturally associated with certain compact Lie groups, as we will outline in Section 4.

3. TOTAL POSITIVITY

In the sequel, we identify the Euclidean space E with $\mathbb{R}^n$. As before, $\Delta \subset \mathbb{R}^n$ is a root system with Weyl group W, and the natural action of $w \in W$ on $t \in \mathbb{R}^n$ is by matrix multiplication, denoted $w \cdot t$. For $t = (t_1, ..., t_n) \in \mathbb{R}^n$ and $w \in W$, we denote by $(w \cdot t)_j$ the jth component of the vector $w \cdot t$. Further, we denote by det w the determinant of $w \in W$ viewed as a linear transformation on $\mathbb{R}^n$, and note that $\det(w) = \pm 1$ since each $w \in W$ is an orthogonal transformation.

The following definition generalizes the alternating sum formula for a classical determinant.

3.1. Definition. Let $\mathscr{D} \subseteq \mathbb{R}^2$ and W be a finite reflection group acting on $\mathbb{R}^n$. For any function $K: \mathscr{D} \to \mathbb{R}$ we define

$$D_W K(s, t) = \sum_{w \in W} (\det w) \prod_{j=1}^{n} K(s_j, (w \cdot t)_j) \tag{3.1}$$

for any $s = (s_1, ..., s_n)$ and $t = (t_1, ..., t_n)$ in $\mathbb{R}^n$ such that $(s_j, (w \cdot t)_j) \in \mathscr{D}$ for all $j = 1, ..., n$ and $w \in W$.

For the case in which $W = \mathfrak{S}_n$, the symmetric group on n symbols, (3.1) reduces to the classical formula for a determinant. For general W, D_W retains some of the properties of the classical determinant. The following proposition, for example, generalizes two familiar determinant properties: (i) If the columns of a determinant are permuted then the sign of the determinant changes in accordance with the sign of the permutation. (ii) If two rows of a determinant are identical then the determinant is identically zero.

3.2. Proposition. (i) *If* $w \in W$, *and* $s, t \in \mathbb{R}^n$, *then* $D_W K(s, w \cdot t) = (\det w)\, D_W K(s, t)$.

(ii) *If t belongs to a wall of a Weyl chamber then* $D_W K(s, t) = 0$.

Proof. (i) Note that $w'w^{-1}$ traverses W as w' traverses W. Then, by (3.1)

$$\begin{aligned}
D_W K(s, w \cdot t) &= \sum_{w' \in W} (\det w') \prod_{j=1}^{n} K(s_j, (w'w \cdot t)_j) \\
&= \sum_{w'w^{-1} \in W} (\det w'w^{-1}) \prod_{j=1}^{n} K(s_j, (w' \cdot t)_j) \\
&= (\det w^{-1}) \sum_{w' \in W} (\det w') \prod_{j=1}^{n} K(s_j, (w' \cdot t)_j) \\
&= (\det w^{-1})\, D_W K(s, t) \\
&= (\det w)\, D_W K(s, t)
\end{aligned}$$

where the last equality holds since w is an orthogonal transformation.

(ii) If t belongs to a wall of a Weyl chamber then there is a reflection $\sigma_\alpha \in W$ that leaves this wall fixed pointwise. Since σ_α is a reflection,

$\det(\sigma_\alpha) = -1$. Therefore $D_W K(s, t) = D_W K(s, \sigma_\alpha \cdot t) = \det(\sigma_\alpha)\, D_W K(s, t) = -D_W K(s, t)$, so $D_W K(s, t) = 0$.

3.3. DEFINITION. Let $\mathfrak{C}$ be the fundamental Weyl chamber. A function $K\colon \mathscr{D} \to \mathbb{R}$ is *totally positive with respect to the group* W if

$$D_W K(s, t) \geqslant 0 \tag{3.2}$$

for any $s = (s_1, ..., s_n)$ and $t = (t_1, ..., t_n)$ in $\mathfrak{C}$ such that $(s_j, (w \cdot t)_j) \in \mathscr{D}$ for all $j = 1, ..., n$ and all $w \in W$. If (3.2) is strictly positive for all $s, t \in \mathfrak{C}$ then we say that K is *strictly totally positive with respect to* W.

We remark in passing that for a given subset $\mathscr{D}$ of $\mathbb{R}^2$ it is possible, even in the classical case in which $W = \mathfrak{S}_n$ and $D_W K$ is the classical determinant, that the domain of $D_W K$ is the empty set. We will assume throughout that $\mathscr{D}$ is such that this pathology does not occur.

Proposition 3.2 describes two properties of classical determinants that are common to D_W for all finite reflection groups W. On the other hand, the invariance of the classical determinant under the interchange of rows and columns does not necessarily extend to finite reflection groups in general. For that reason, we make the following definition.

3.4 DEFINITION. We say that the function K is *W-symmetric* if whenever $D_W K$ is defined at (s, t) it is also defined at $(w \cdot s, t)$ for all $w \in W$, and

$$D_W K(s, w \cdot t) = D_W K(w \cdot s, t).$$

We will also say that a nonnegative Borel measure μ on $\mathbb{R}$ is *invariant under* W if the product measure $d\mu(s_1) \cdots d\mu(s_n)$, $s = (s_1, ..., s_n) \in \mathbb{R}^n$, is invariant under the action of W on $\mathbb{R}^n$. Examples of W-invariant measures are Lebesgue measure and the Gaussian measure $d\mu(x) = \exp(-x^2)\, dx$. These measures are W-invariant since the resulting product measures on $\mathbb{R}^n$ are invariant under all orthogonal transformations.

We next generalize the basic composition formula [12, p. 17] to the setting of finite reflection groups.

3.5. THEOREM. *Let L and M be W-symmetric on $\mathbb{R}^2$. Suppose that both L and M are totally positive with respect to W, let μ be a W-invariant measure on $\mathbb{R}$, and define*

$$K(x, y) = \int_{\mathbb{R}} L(x, z)\, M(z, y)\, d\mu(z) \tag{3.3}$$

for $(x, y) \in \mathbb{R}^2$, whenever the integral (3.3) *converges absolutely. Then K is W-symmetric and totally positive.*

Proof. By definition, for $s=(s_1, ..., s_n)$ and $t=(t_1, ..., t_n)$,

$$D_W K(s, t) = \sum_W (\det w) \prod_{j=1}^{n} K(s_j, (w \cdot t)_j)$$
$$= \sum_W (\det w) \prod_{j=1}^{n} \int_{\mathbb{R}} L(s_j, z_j)\, M(z_j, (w \cdot t)_j)\, d\mu(z_j).$$

Interchanging the sum and integral, and recalling the definition of $D_W M$, we obtain

$$D_W K(s, t) = \int_{\mathbb{R}^n} \left[\prod_{j=1}^{n} L(s_j, z_j) \right] D_W M(\zeta, t) \prod_{j=1}^{n} d\mu(z_j) \tag{3.4}$$

where $\zeta=(z_1, ..., z_n)$. Since $D_W M(\zeta, t)=0$ if ζ belongs to the wall P_α of a Weyl chamber (Proposition 3.2 (ii)), (3.4) reduces to

$$D_W K(s, t) = \int_{\mathbb{R}^n \setminus \bigcup_\alpha P_\alpha} \left[\prod_{j=1}^{n} L(s_j, z_j) \right] D_W M(\zeta, t) \prod_{j=1}^{n} d\mu(z_j).$$

Writing $\mathbb{R}^n \setminus \bigcup_\alpha P_\alpha$ as a disjoint union of Weyl chambers,

$$\mathbb{R}^n \Big\backslash \bigcup_\alpha P_\alpha = \bigcup_{w \in W} w \cdot \mathfrak{C}$$

we obtain

$$D_W K(s, t) = \sum_{w \in W} \int_{\zeta \in w \cdot \mathfrak{C}} \left[\prod_{j=1}^{n} L(s_j, z_j) \right] D_W M(\zeta, t) \prod_{j=1}^{n} d\mu(z_j)$$
$$= \sum_{w \in W} \int_{w \cdot \zeta \in w \cdot \mathfrak{C}} \left[\prod_{j=1}^{n} L(s_j, (w \cdot \zeta)_j) \right] D_W M(w \cdot \zeta, t) \prod_{j=1}^{n} d\mu((w \cdot \zeta)_j).$$

Since $D_W M(w \cdot \zeta, t) = D_W M(\zeta, w \cdot t) = (\det w) D_W M(\zeta, t)$ and $\prod_{j=1}^{n} d\mu((w \cdot \zeta)_j) = \prod_{j=1}^{n} d\mu(z_j)$, we obtain

$$D_W K(s, t) = \int_{\mathfrak{C}} \left[\sum_W (\det w) \prod_{j=1}^{n} L(s_j, (w \cdot \zeta)_j) \right] D_W M(\zeta, t) \prod_{j=1}^{n} d\mu(z_j)$$
$$= \int_{\mathfrak{C}} D_W L(s, \zeta)\, D_W M(\zeta, t) \prod_{j=1}^{n} d\mu(z_j).$$

It is now evident that if L and M are W-symmetric and totally positive with respect to W, then K is also W-symmetric and totally positive with respect to W.

3.6. *Remarks.* (a) The last equation above,

$$D_W K(s, t) = \int_{\mathfrak{C}} D_W L(s, \zeta)\, D_W M(\zeta, t) \prod_{j=1}^{n} d\mu(z_j) \tag{3.5}$$

is a generalization for finite reflection groups of the classical Binet–Cauchy formula. Indeed, (3.5) is representative of a general group-theoretic construction. If one replaces the determinant in (3.1) by any one-dimensional representation χ of W then (3.1) becomes

$$D_{W,\chi} K(s, t) = \sum_{w \in W} \chi(w) \prod_{j=1}^{n} K(s_j, (w \cdot t)_j).$$

The function $D_{W,\chi}$ satisfies the *χ-covariance property*,

$$D_{W,\chi} K(s, w \cdot t) = \chi(w)\, D_{W,\chi} K(s, t)$$

for $s, t \in \mathbb{R}^n$ and $w \in W$, and the corresponding Binet–Cauchy formula

$$D_{W,\chi} K(s, t) = \int_{\mathfrak{C}} D_{W,\chi} L(s, \zeta)\, D_{W,\chi} M(\zeta, t) \prod_{j=1}^{n} d\mu(z_j) \tag{3.6}$$

holds, where K is given by (3.3). In particular, for $\chi \equiv 1$, (3.6) generalizes the Binet–Cauchy formula for the permanent. Observations similar to these have been made by Karlin and Rinott [14] and Stembridge [17] in their generalizations of the Binet–Cauchy formula.

(b) The hypotheses of Theorem 3.5 can be relaxed as follows: Suppose L is defined on $\mathscr{D}_L = \mathscr{D}_1 \times \mathscr{D}_2$ and M is defined on $\mathscr{D}_M = \mathscr{D}_2 \times \mathscr{D}_3$ where $\mathscr{D}_1$, $\mathscr{D}_2$ and $\mathscr{D}_3$ are subsets of $\mathbb{R}$. Denote by $\mathscr{D}_2^{(n)}$ the Cartesian product $\mathscr{D}_2 \times \cdots \mathscr{D}_2$ (n factors), and assume that the subset $\mathscr{D}_2^{(n)}$ of $\mathbb{R}^n$ is invariant under W. Theorem 3.5 now holds with $\mathscr{D}_2$ as the domain of integration in (3.3) and $\mathscr{D}_2^{(n)} \cap \mathfrak{C}$ as the domain of integration in (3.5).

4. A Formula of Harish-Chandra

We now turn our attention to an integral formula of Harish-Chandra from which we will generate examples of totally positive functions for certain finite reflection groups W.

By way of motivation for this formula, we first review the proof in [7] that the function $K(x, y) = e^{xy}$ on $\mathbb{R}^2$ is totally positive in the classical sense. For this function K, formula (1.2) takes the form

$$V(s)\, V(t) \int_{\mathrm{U}(n)} e^{\operatorname{tr} usu^{-1}t}\, du = \beta_n \det(e^{s_j t_k}) \tag{4.1}$$

where

$$\beta_n = \prod_{j=1}^{n} (j-1)! \tag{4.2}$$

By expressing $\det(e^{s_j t_k})$ as an alternating sum over the symmetric group $\mathfrak{S}_n$, we can rewrite (4.1) in the form

$$\int_{U(n)} e^{\langle \mathrm{Ad}(u) \cdot s | t \rangle} \, du = \beta_n \frac{\sum_{w \in \mathfrak{S}_n} \varepsilon(w) \, e^{\langle w \cdot s | t \rangle}}{V(s) \, V(t)} \tag{4.3}$$

where $\mathrm{Ad}(u) \cdot s = usu^{-1}$ is the *adjoint action* of $u \in \mathrm{U}(n)$ on $s = \mathrm{diag}(s_1, ..., s_n)$; $\varepsilon(w)$ is the sign of the permutation w; $\langle s|t \rangle = \mathrm{tr}(st)$; and $w \cdot s = \mathrm{diag}(s_{w(1)}, ..., s_{w(n)})$. It follows immediately from (4.3) that the alternating sum on the right-hand side of (4.3) is positive for all s, t. Equivalently, the function $K(x, y) = e^{xy}$ on $\mathbb{R}^2$ is totally positive in the classical sense.

As we now explain, (4.3) is a special case of Harish–Chandra's formula for compact Lie groups. Since we do not assume that the reader is familiar with Lie theory, we will provide enough background to understand the ingredients in the formula. For a complete treatment of the structure theory of compact Lie groups—and especially the interplay of compact groups and root systems—the reader should consult a reference on Lie groups, for instance, [3, 9, 15, 18].

Let U be a compact connected Lie group. Without loss of generality U can be taken to be a closed, bounded, and connected subgroup of the general linear group $GL(N, \mathbb{C})$ of complex nonsingular $N \times N$ matrices, for some value of N. Denote by $\mathbb{C}^{N \times N}$ the space of complex $N \times N$ matrices. A Lie algebra can then be thought of as a subspace of $\mathbb{C}^{N \times N}$ that is closed under commutators $[X, Y] = XY - YX$. The Lie algebra is real or complex according to whether it is closed under real or complex scalars, respectively. In particular, the Lie algebra $\mathfrak{u}$ of U is the collection of all $N \times N$ matrices X such that

$$e^{tX} = \sum_{j=0}^{\infty} \frac{t^j}{j!} X^j$$

is in U for all $t \in \mathbb{R}$. The exponential map $X \mapsto e^X$, while not one-to-one, maps $\mathfrak{u}$ onto U.

Given $u \in U$ we define the linear transformation $\mathrm{Ad}(u)$ on $\mathfrak{u}$ by

$$\mathrm{Ad}(u) \cdot Y = uYu^{-1} \tag{4.4}$$

for $Y \in \mathfrak{u}$. The mapping $u \mapsto \mathrm{Ad}(u)$ is called the adjoint representation of U. Correspondingly, the formula

$$\mathrm{ad}(X)\, Y = [X, Y] \tag{4.5}$$

for $X, Y \in \mathfrak{u}$ defines a linear transformation $\mathrm{ad}(X)$ on $\mathfrak{u}$, and the mapping $X \mapsto \mathrm{ad}(X)$ is called the adjoint representation of $\mathfrak{u}$. The representations ad of $\mathfrak{u}$ and Ad of U are related by the formula

$$\mathrm{Ad}(e^{tX}) = e^{t\,\mathrm{ad}(X)}$$

for all $X \in \mathfrak{u}$ and $t \in \mathbb{R}$.

Define the complex vector space $\mathfrak{g}$ by

$$\mathfrak{g} = \mathfrak{u} + i\mathfrak{u} = \{X + iY \in \mathbb{C}^{N \times N} \colon X, Y \in \mathfrak{u}\} \tag{4.6}$$

where $i = \sqrt{-1}$. Then $\mathfrak{g}$ is a complex Lie algebra called the *complexification* of $\mathfrak{u}$. For each $u \in U$, formula (4.4) makes sense for $Y \in \mathfrak{g}$, $\mathrm{Ad}(u)$ becomes a linear transformation on $\mathfrak{g}$, and the mapping $u \mapsto \mathrm{Ad}(u)$ becomes the adjoint representation of U acting on $\mathfrak{g}$. Similarly, formula (4.5) is defined for $X, Y \in \mathfrak{g}$, and the mapping $X \mapsto \mathrm{ad}(X)$ becomes the adjoint representation of $\mathfrak{g}$.

A Lie algebra is *simple* if it has no nontrivial ideals, is *semisimple* if it is the direct sum of its simple ideals, and is *reductive* if it is the direct sum of its center and a semisimple ideal. Correspondingly, a Lie group is simple, semisimple, or reductive if its Lie algebra is simple, semisimple, or reductive, respectively.

A compact Lie group U is always reductive. Indeed, its Lie algebra $\mathfrak{u}$ is the direct sum of the center $\mathfrak{z}$ and the semisimple ideal $[\mathfrak{u}, \mathfrak{u}]$ generated by all commutators. At the Lie group level, this splitting translates into the following: There exists a semisimple subgroup U_0 of U such that each $u \in U$ can be written

$$u = zu_0 \tag{4.7}$$

where $z \in Z$, the center of U, and $u_0 \in U_0$. For example, the unitary group $\mathrm{U}(n)$ has a one-dimensional center consisting of the scalar multiples $e^{i\theta}I$ where I is the identity matrix, and so $\mathrm{U}(n)$ is reductive; the special unitary group $\mathrm{SU}(n)$, consisting of those unitary matrices having determinant 1, is simple; and $\mathrm{SU}(n)$ is the semisimple (in this case simple) part of $\mathrm{U}(n)$. Thus, the general element u of $\mathrm{U}(n)$ is of the form $u = e^{i\theta}u_0$ where $u_0 \in \mathrm{SU}(n)$ and $\theta \in [0, 2\pi)$.

A connected abelian subgroup of a compact Lie group U is called a *torus*. Central to the structure theory for compact Lie groups is the existence, and uniqueness up to isomorphism, of a maximal such subgroup.

Thus, let $T \subset U$ be a *maximal torus* of U, which is necessarily isomorphic to a product of one-dimensional unitary groups $U(1)$. Then T is a maximal Abelian subgroup of U, and its Lie algebra $\mathfrak{t}$ is a maximal Abelian subalgebra of $\mathfrak{u}$. Denote by $\mathfrak{t}_0$ the semisimple part of $\mathfrak{t}$. That is, $\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{t}_0$ where $\mathfrak{z}$ is the center of $\mathfrak{u}$ and $\mathfrak{t}_0 = \mathfrak{t} \cap [\mathfrak{u}, \mathfrak{u}]$.

Let $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$ be the complexification of $\mathfrak{t}$. Then $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$, the *Cartan subalgebra*, such that the linear transformations $\mathrm{ad}(H)$ for $H \in \mathfrak{h}$ are simultaneously diagonalizable. Let $\mathfrak{h}^*$ denote the dual space of $\mathfrak{h}$. For $\alpha \in \mathfrak{h}^*$ we set

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g}: [H, X] = \alpha(H)\, X \text{ for all } H \in \mathfrak{h}\}.$$

If $\mathfrak{g}^\alpha$ is nonzero we say that α is a *root* of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and that $\mathfrak{g}^\alpha$ is the corresponding *root space*. We denote the set of roots by Δ. Then each root space is one-dimensional, $\mathfrak{g}^0 = \mathfrak{h}$, and $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \oplus \mathfrak{g}^\alpha. \tag{4.8}$$

Equation (4.8) is called the *root space decomposition* of $\mathfrak{g}$.

The next element of structure is the existence of a non-degenerate complex bilinear form $\langle \cdot | \cdot \rangle$ on $\mathfrak{g}$ which is $\mathfrak{g}$-invariant in the sense that $\langle [X, Y] | Z \rangle = -\langle Y | [X, Z] \rangle$ for all $X, Y, Z \in \mathfrak{g}$. From the invariance and the root space decomposition, it follows that the restriction of this form to $\mathfrak{h}$ is non-degenerate. Moreover, the restriction of this form to the real subspace $i\mathfrak{t}$ of $\mathfrak{h}$ is an inner product on $i\mathfrak{t}$. In particular, for each $\alpha \in \Delta$ there exists a unique element $H_\alpha \in i\mathfrak{t}_0$ such that $\alpha(H) = \langle H | H_\alpha \rangle$ for all $H \in i\mathfrak{t}$.

For our purposes, what is most crucial is the following. If we lift the inner product from $i\mathfrak{t}_0$ to the dual space $i\mathfrak{t}_0^*$, then $E = i\mathfrak{t}_0^*$ becomes a Euclidean space in which the subset Δ of roots is an abstract root system in the sense of Definition 2.1, and the theory described in Section 2 may be applied. The Weyl group W for this root system is called the *Weyl group of the compact group* U. In particular, we choose an ordering for the roots, specify a system Δ_+ of positive roots, and define a fundamental Weyl chamber $\mathfrak{C}$.

Now we can state Harish-Chandra's formula.

4.1. THEOREM (Harish-Chandra). For $H_1, H_2 \in \mathfrak{h}$

$$\int_U e^{\langle \mathrm{Ad}(u) \cdot H_1 | H_2 \rangle}\, du = \beta_W \frac{\sum_{w \in W} (\det w)\, e^{\langle w \cdot H_1 | H_2 \rangle}}{V(H_1)\, V(H_2)} \tag{4.9}$$

where β_W *is a positive constant, and*

$$V(H) = \prod_{\alpha \in \Delta_+} \alpha(H)$$

for any $H \in \mathfrak{h}$.

Note that there is no loss of generality if in formula (4.9) we take U to be semisimple. The reason is as follows: For any $z \in Z$, $\mathrm{Ad}(z)$ is the identity. Thus if we write $u \in U$ according to (4.7) as $u = zu_0$ then $\mathrm{Ad}(u) = \mathrm{Ad}(z)\,\mathrm{Ad}(u_0) = \mathrm{Ad}(u_0)$. In this way we can replace the integration over U in (4.9) by integration over U_0 and all that changes is the value of the constant β_W. We also remark that the constant β_W can be computed explicitly by several methods. However for our purposes it is sufficient to know that β_W is positive.

For a proof of Theorem 4.1, we refer the reader to Harish-Chandra's original paper [8, Theorem 2, p. 104] or Helgason [9, p. 329].

5. The Unitary and Orthogonal Groups

In this section, we make Theorem 4.1 explicit both for the unitary groups $\mathrm{U}(n)$ and the special orthogonal groups $\mathrm{SO}(n)$.

5.1. *The Unitary Groups.* From the remarks at the end of the preceding section it suffices to consider the subgroup $\mathrm{SU}(n)$ rather than $\mathrm{U}(n)$ itself. Since $\mathrm{SU}(1)$ reduces to the identity, we assume that $n \geqslant 2$. Therefore, we set $U = \mathrm{SU}(n)$, and note that its Lie algebra is $\mathfrak{u} = \mathfrak{su}(n)$, the Lie algebra of all $n \times n$ complex skew-Hermitian matrices with trace zero.

The complexification of $\mathfrak{u}$ is $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, the Lie algebra of all $n \times n$ complex matrices with trace zero, on which an invariant bilinear form is given by $\langle X \mid Y \rangle = \mathrm{tr}(XY)$ for $X, Y \in \mathfrak{g}$. As maximal torus T in U we take the subgroup of diagonal matrices $\mathrm{diag}(e^{i\theta_1}, ..., e^{i\theta_n})$ where $\theta_1, ..., \theta_n \in \mathbb{R}$ and $\theta_1 + \cdots + \theta_n = 0$. Then the Lie algebra $\mathfrak{t}$ of T consists of all diagonal matrices $\mathrm{diag}(i\theta_1, ..., i\theta_n)$ with $\theta_1 + \cdots + \theta_n = 0$; $\mathfrak{t}_0 = \mathfrak{t}$; and the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ consists of all diagonal $n \times n$ complex matrices $H = \mathrm{diag}(h_1, ..., h_n)$ such that $h_1 + \cdots + h_n = 0$.

Define the linear functional e_j on $\mathfrak{h}$ by $e_j(H) = h_j$, where $H = \mathrm{diag}(h_1, ..., h_n)$. Also let E_{jk} be the $n \times n$ matrix with 1 in the (j, k)th position and 0 elsewhere. Then the linear functional $\alpha = e_j - e_k$, $j \neq k$, is a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$, the corresponding root space is $\mathfrak{g}^\alpha = \mathbb{C}E_{jk}$, and these linear functionals α exhaust all of the roots. That is, $\Delta = \{e_j - e_k : 1 \leqslant j \neq k \leqslant n\}$. If we identify the linear functionals $e_1, ..., e_n$ with the standard basis of $\mathbb{R}^n$, then Δ is precisely the root system encountered earlier in Example 2.2 and Harish-Chandra's formula (4.9) for $\mathrm{SU}(n)$ (or equivalently $\mathrm{U}(n)$) is precisely (4.3).

5.2. *The Orthogonal Groups.* We next examine Harish–Chandra's formula for the orthogonal group $O(N)$ of real $N \times N$ matrices u such that $uu' = 1$. Since in Theorem 4.1 we require a compact group to be connected, we restrict attention to the special orthogonal subgroup $SO(N)$, consisting of those orthogonal matrices having determinant one. Since the group $SO(2)$ is abelian, we assume that $N \geqslant 3$.

Let $U = SO(N)$. The Lie algebra of U is $\mathfrak{u} = \mathfrak{so}(N)$, the Lie algebra of all $N \times N$ real skew-symmetric matrices with trace zero. The complexification of $\mathfrak{u}$ is $\mathfrak{g} = \mathfrak{so}(N, \mathbb{C})$, the Lie algebra of all $N \times N$ complex skew-symmetric matrices with trace zero. An invariant bilinear form on $\mathfrak{g}$ is given by $\langle X \mid Y \rangle = \operatorname{tr}(XY)$.

We remark in passing that the Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic, as are the pairs $\mathfrak{so}(4)$ and $\mathfrak{su}(2) \times \mathfrak{su}(2)$ as well as $\mathfrak{so}(6)$ and $\mathfrak{su}(3)$. In each case the corresponding compact groups are locally isomorphic, and locally isomorphic groups give rise to the same notion of total positivity. In what follows we consider the cases N even and N odd separately.

5.3. *The Group* $SO(N)$, $N = 2n$ *Even.* For the maximal torus T in $U = SO(2n)$ we take the group of all $2n \times 2n$ block-diagonal matrices of the form

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & & & \\ -\sin\theta_1 & \cos\theta_1 & & & & & \\ & & \cos\theta_2 & \sin\theta_2 & & & \\ & & -\sin\theta_2 & \cos\theta_2 & & & \\ & & & & \ddots & & \\ & & & & & \cos\theta_n & \sin\theta_n \\ & & & & & -\sin\theta_n & \cos\theta_n \end{pmatrix}$$

where $\theta_1, ..., \theta_n \in \mathbb{R}$. Then the Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ is the complex Lie algebra of all $2n \times 2n$ complex block-diagonal matrices of the form

$$H = \begin{pmatrix} 0 & h_1 & & & & & \\ -h_1 & 0 & & & & & \\ & & 0 & h_2 & & & \\ & & -h_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & h_n \\ & & & & & -h_n & 0 \end{pmatrix} \tag{5.1}$$

where $h_1, ..., h_n \in \mathbb{C}$.

As in the case of the unitary groups, we could write out the root space decomposition in detail; but since the computations, although routine, are somewhat more lengthy than in the unitary case, we refer the interested reader to Knapp [15, Ch. 4, p. 60 ff]. We will write down the root system, the Weyl group, and the fundamental Weyl chamber as follows.

Let $\{e_1, ..., e_n\}$ be the standard basis for $\mathbb{R}^n$. A root system for SO(2n) is $\Delta = \{\pm e_j \pm e_k : 1 \leq j < k \leq n\}$. In the general classification for root systems, Δ is a root system of type D_n.

A system of positive roots for SO(2n) is $\Delta_+ = \{e_j \pm e_k : 1 \leq j < k \leq n\}$. For $\alpha = e_j \pm e_k \in \Delta_+$ and H of the form (5.1), $\alpha(H) = h_j \pm h_k$ so that

$$V(H) = \prod_{1 \leq j < k \leq n} (h_j - h_k)(h_j + h_k) = \prod_{1 \leq j < k \leq n} (h_j^2 - h_k^2).$$

Also, a fundamental Weyl chamber is $\mathfrak{C} = \{(h_1, ..., h_n) \in \mathbb{R}^n : h_1 > h_2 > \cdots > h_{n-1} > |h_n|\}$.

Let G(n) denote the group of permutations w of the set $\{-n, ..., -1, 1, ..., n\}$ for which $w(-j) = -w(j)$. If we let $h_1, ..., h_n$ be n symbols and define $h_{-j} = -h_j$, $j = 1, ..., n$, then G(n) acts on the set of symbols $\{h_{-n}, ..., h_{-1}, h_1, ..., h_n\}$ by

$$w \cdot (h_1, ..., h_n) = (h_{w(1)}, ..., h_{w(n)}).$$

The Weyl group of SO(2n) is SG(n), the subgroup of G(n) consisting of even permutations [3, p. 171]. Put more simply, SG(n) is the group generated by the permutation group $\mathfrak{S}_n$ together with an even number of sign changes.

The action of G(n) carries over in a natural way to the Lie algebra $\mathfrak{h}$. Let

$$b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so that the matrix H in (5.3) may be written as the direct sum

$$H = h_1 b \oplus h_2 b \oplus \cdots \oplus h_n b.$$

Then the action of G(n) on $\mathfrak{h}$ is given by

$$w \cdot H = h_{w(1)} b \oplus h_{w(2)} b \oplus \cdots \oplus h_{w(n)} b \tag{5.2}$$

for $w \in$ G(n) and $H \in \mathfrak{h}$.

With the above ingredients now in place, Harish-Chandra's formula for SO(2n) is as follows: If $H_1 = h_{1,1}b \oplus h_{1,2}b \oplus \cdots \oplus h_{1,n}b$ and $H_2 = h_{2,1}b \oplus h_{2,2}b \oplus \cdots \oplus h_{2,n}b$, then

$$\int_{SO(2n)} e^{\operatorname{tr}(uH_1u^{-1}H_2)}\,du = \beta_{SG(n)} \frac{\sum_{w \in SG(n)} (\det w)\, e^{\langle w \cdot H_1 \mid H_2 \rangle}}{V(H_1)\, V(H_2)}$$

$$= \beta_{SG(n)} \frac{\sum_{w \in SG(n)} (\det w) \exp(2 \sum_{j=1}^{n} w(h_{1,j})\, h_{2,j})}{\prod_{1 \leqslant j < k \leqslant n} (h_{1,j}^2 - h_{1,k}^2)(h_{2,j}^2 - h_{2,k}^2)}$$

and the value of the constant $\beta_{SG(n)}$ can be calculated to be

$$\beta_{SG(n)} = (2n-1)! \prod_{j=1}^{2n-1} (2j-1)!$$

5.4. *The Group* SO(N), $N = 2n+1$ *odd.* Now let $U = SO(2n+1)$. A maximal torus for U is the group T of all $(2n+1) \times (2n+1)$ block-diagonal matrices of the form

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & & & & \\ -\sin\theta_1 & \cos\theta_1 & & & & & & \\ & & \cos\theta_2 & \sin\theta_2 & & & & \\ & & -\sin\theta_2 & \cos\theta_2 & & & & \\ & & & & \ddots & & & \\ & & & & & \cos\theta_n & \sin\theta_n & \\ & & & & & -\sin\theta_n & \cos\theta_n & \\ & & & & & & & 1 \end{pmatrix}$$

where $\theta_1, ..., \theta_n \in \mathbb{R}$. The complexification $\mathfrak{h}$ of $\mathfrak{t}$ is the Lie algebra of all $(2n+1) \times (2n+1)$ block-diagonal matrices of the form

$$H = \begin{pmatrix} 0 & h_1 & & & & & & \\ -h_1 & 0 & & & & & & \\ & & 0 & h_2 & & & & \\ & & -h_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & h_n & \\ & & & & & -h_n & 0 & \\ & & & & & & & 0 \end{pmatrix} \tag{5.3}$$

where $h_1, ..., h_n \in \mathbb{C}$. A root system for $SO(2n+1)$ is

$$\Delta = \{\pm e_j \pm e_k : 1 \leqslant j < k \leqslant n\} \cup \{\pm e_j : 1 \leqslant j \leqslant n\}$$

and a system of positive roots is

$$\Delta_+ = \{e_j \pm e_k : 1 \leqslant j < k \leqslant n\} \cup \{e_j : 1 \leqslant j \leqslant n\}.$$

For $\alpha \in \Delta_+$ and H of the form (5.3)

$$\alpha(H) = \begin{cases} h_j \pm h_k & \text{if } \alpha = e_j \pm e_k \\ h_j & \text{if } \alpha = e_j. \end{cases}$$

Then

$$V(H) = \prod_{1 \leqslant j < k \leqslant n} (h_j^2 - h_k^2) \prod_{j=1}^{n} h_j.$$

Moreover, a fundamental Weyl chamber is $\mathfrak{C} = \{(h_1, ..., h_n) \in \mathbb{R}^n : h_1 > \cdots > h_n > 0\}$.

The Weyl group of SO$(2n+1)$ is $W = \mathrm{G}(n)$, the reflection group described earlier, and the action of $\mathrm{G}(n)$ on the Lie algebra $\mathfrak{h}$, in analogy with (5.2), is given by

$$w \cdot H = h_{w(1)} b \oplus h_{w(2)} b \oplus h_{w(n)} b \oplus 0$$

for H of the form (5.3). Then, Harish–Chandra's formula for SO$(2n+1)$ is as follows: If $H_1 = h_{1,1} b \oplus h_{1,2} b \oplus \cdots \oplus h_{1,n} b \oplus 0$ and $H_2 = h_{2,1} b \oplus h_{2,2} b \oplus \cdots \oplus h_{2,n} b \oplus 0$, then

$$\int_{\mathrm{SO}(2n+1)} e^{\mathrm{tr}(uH_1u^{-1}H_2)}\, du$$

$$= \beta_{\mathrm{G}(n)} \frac{\sum_{w \in \mathrm{G}(n)} (\det w)\, e^{\langle w \cdot H_1 \mid H_2 \rangle}}{V(H_1)\, V(H_2)}$$

$$= \beta_{\mathrm{G}(n)} \frac{\sum_{w \in \mathrm{G}(n)} (\det w) \exp(2 \sum_{j=1}^{n} w(h_{1,j})\, h_{2,j})}{\prod_{1 \leqslant j < k \leqslant n} (h_{1,j}^2 - h_{1,k}^2)(h_{2,j}^2 - h_{2,k}^2) \prod_{j=1}^{n} h_{1,j} h_{2,j}}$$

and the constant $\beta_{\mathrm{G}(n)}$ is

$$\beta_{\mathrm{G}(n)} = \prod_{j=1}^{n} (2j-1)! \prod_{j=2n}^{4n-1} j!$$

6. Examples of Totally Positive Functions

Using Harish–Chandra's formula, we obtain the first example of a function that is totally positive function with respect to W.

6.1. THEOREM. *Suppose that W is the Weyl group for a compact Lie group U. Then the function $K(x, y) = e^{xy}$, $(x, y) \in \mathbb{R}^2$, is strictly totally positive with respect to W.*

Proof. With notation as in Section 4, suppose that the rank of the root system Δ is n. Then the real vector space it_0 has dimension n, and relative to a choice of basis we can identify it_0 with $\mathbb{R}^n$. Thus, we identify elements H_1 and H_2 of it_0 with the vectors $s = (s_1, ..., s_n)$ and $t = (t_1, ..., t_n)$, respectively, in $\mathbb{R}^n$. Now suppose s and t are in the fundamental Weyl chamber $\mathfrak{C}$. By Harish-Chandra's theorem and the definition of $D_W K$, we have

$$\begin{aligned} D_W K(s, t) &= \sum_{w \in W} (\det w) \prod_{j=1}^{n} e^{(w \cdot s)_j t_j} \\ &= \sum_{w \in W} (\det w)\, e^{\langle w \cdot H_1 \mid H_2 \rangle} \\ &= \beta_W^{-1} V(H_1)\, V(H_2) \int_U e^{\langle \mathrm{Ad}(u) \cdot H_1 \mid H_2 \rangle}\, du. \end{aligned}$$

Since $s, t \in \mathfrak{C}$, it follows from the definition of $V(H)$ that $V(H_1) > 0$, $V(H_2) > 0$. Also, $e^{\langle \mathrm{Ad}(u) \cdot H_1 \mid H_2 \rangle} > 0$, since $\langle \mathrm{Ad}(u) \cdot H_1 \mid H_2 \rangle$ is real for all $u \in U$. Hence $D_W K(s, t) > 0$ for all $s, t \in \mathfrak{C}$, which is the statement that K is strictly totally positive with respect to W.

If W is the Weyl group of the compact Lie group U and the function $K: \mathbb{R}^2 \to \mathbb{R}$ is totally positive with respect to W then we will simply say that K is totally positive with respect to U. Thus, a function K is totally positive of order r in the classical sense if and only if K is totally positive with respect to all the groups $\mathrm{U}(1), \mathrm{U}(2), ..., \mathrm{U}(r)$.

We now construct examples of functions that are totally positive with respect to the special orthogonal groups. In the case in which $\mathrm{U} = \mathrm{SO}(2n+1)$, recall that the elements of the Weyl group $W = \mathrm{G}(n)$ can be viewed as permutations of the set $\{1, ..., n\}$ together with up to n possible sign changes. Therefore, each $w \in W$ is of the form $w = \varepsilon\sigma$, where $\sigma \in \mathfrak{S}_n$ is a permutation on n symbols, $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$ with $\varepsilon_j = \pm 1$ for all $j = 1, ..., n$, and $w = \varepsilon\sigma$ acts on $s = (s_1, ..., s_n) \in \mathbb{R}^n$ by $w \cdot s = (\varepsilon_1 s_{\sigma(1)}, ..., \varepsilon_n s_{\sigma(n)})$. Writing out the sum in (3.1) with $W = \mathrm{G}(n)$, we have

$$\begin{aligned} D_W K(s, t) &= \sum_{\varepsilon_1 = \pm 1, ..., \varepsilon_n = \pm 1} \left(\prod_{j=1}^{n} \varepsilon_j \right) \sum_{\sigma \in \mathfrak{S}_n} (\det \sigma) \prod_{j=1}^{n} K(s_j, \varepsilon_{\sigma(j)} t_{\sigma(j)}) \\ &= \sum_{\varepsilon_1 = \pm 1, ..., \varepsilon_n = \pm 1} \left(\prod_{j=1}^{n} \varepsilon_j \right) \begin{vmatrix} K(s_1, \varepsilon_1 t_1) & K(s_1, \varepsilon_2 t_2) & \cdots & K(s_1, \varepsilon_n t_n) \\ K(s_2, \varepsilon_1 t_1) & K(s_2, \varepsilon_2 t_2) & \cdots & K(s_2, \varepsilon_n t_n) \\ \vdots & \vdots & & \vdots \\ K(s_n, \varepsilon_1 t_1) & K(s_n, \varepsilon_2 t_2) & \cdots & K(s_n, \varepsilon_n, t_n) \end{vmatrix}. \end{aligned} \tag{6.1}$$

Thus, the function K is totally positive with respect to $SO(2n+1)$ if (6.1) is nonnegative for all $s_1 > \cdots > s_n > 0$ and $t_1 > \cdots > t_n > 0$.

As an example, substituting $K(x, y) = e^{xy}$, $x, y \in \mathbb{R}$, in (6.1), we obtain

$$\begin{aligned} D_W K(s, t) &= \sum_{\sigma \in \mathfrak{S}_n} \det(\sigma) \sum_{\varepsilon_1 = \pm 1, \ldots, \varepsilon_n = \pm 1} \prod_{j=1}^{n} \varepsilon_j e^{s_{\sigma(j)} \varepsilon_j t_j} \\ &= \sum_{\sigma \in \mathfrak{S}_n} (\det \sigma) \prod_{j=1}^{n} (e^{s_{\sigma(j)} t_j} - e^{-s_{\sigma(j)} t_j}) \\ &= \det(e^{s_j t_k} - e^{-s_j t_k}) \\ &= 2^n \det(\sinh(s_j t_k)). \end{aligned}$$

Therefore, the strict total positivity of K with respect to $SO(2n+1)$ is equivalent to the well-known result that the function $L(x, y) = \sinh(xy)$, $x, y \in \mathbb{R}_+$, is strictly totally positive in the classical sense [12]. In the case in which $n = 2$ the strict total positivity of K with respect to $SO(5)$, as expressed through the positivity of (6.2), reduces to the inequality (1.3).

Let $\mathscr{K}_0$ denote the class of even functions $K: \mathbb{R}^2 \to \mathbb{R}$; that is, $K(-x, -y) = K(x, y)$. If $K \in \mathscr{K}_0$ then it follows from (6.1) that $D_W K(s, w \cdot t) = D_W K(w \cdot s, t)$ for all $w \in G(n)$ and $s, t \in \mathbb{R}^n$; that is, K is W-symmetric in the sense of Definition 3.4.

Let μ be a positive Borel measure on $\mathbb{R}$. Then we can show that μ is $G(n)$-invariant if and only if μ is even; that is, $d\mu(-z) = d\mu(z)$. To prove this, note that μ is $G(n)$-invariant if and only if

$$\int_A d\mu(z_1) \cdots d\mu(z_n) = \int_{w \cdot A} d\mu(z_1) \cdots d\mu(z_n)$$

for any $w \in G(n)$ and any Borel set $A \subset \mathbb{R}^n$. Choosing A to be a Cartesian product of intervals, it follows that $\mu(I) = \mu(-I)$ for any interval $I \subset \mathbb{R}$. Hence, $d\mu(-z) = d\mu(z)$.

We conclude from the above remarks that the basic composition formula, Theorem 3.5, holds in the case of $SO(2n+1)$ for all $L, M \in \mathscr{K}_0$ and any even measure μ such that the integral in (3.3) converges absolutely.

6.2. EXAMPLE. Let $\mathscr{D} = \{(x, y) \in \mathbb{R}^2 : |x + y| < 1\}$ and $K: \mathscr{D} \to \mathbb{R}$ where $K(x, y) = [1 - (x + y)^2]^{-1}$. Then K is totally positive with respect to $SO(2n+1)$.

Proof. This follows from the basic composition formula with $L(x, y) = M(x, y) = e^{xy}$ and $d\mu(x) = \frac{1}{2} e^{-|x|}\, dx$.

Note that K is also totally positive with respect to $U(n)$, that is, TP_n in the classical sense. This holds since L, M are totally positive with respect to $U(n)$ and since any positive Borel measure μ is $\mathfrak{S}_n$-invariant.

6.3. *Remarks*. Suppose that K is totally positive with respect to $\mathrm{SO}(2n+1)$, and $f, g\colon \mathbb{R} \to \mathbb{R}_+$ are arbitrary functions. Then the function $\tilde{K}(x, y) = f(|x|)\, g(|y|)\, K(x, y)$ is also totally positive with respect to $\mathrm{SO}(2n+1)$. This result follows directly from (6.1).

In particular, by choosing $K(x, y) = e^{2xy}$, $f(x) = g(x) = \exp(x^2)$, we deduce that the function $\tilde{K}(x, y) = \exp((x+y)^2)$, $(x, y) \in \mathbb{R}^2$, is totally positive with respect to $\mathrm{SO}(2n+1)$.

Next we turn to the case in which $U = \mathrm{SO}(2n)$. Now the Weyl group becomes $W = \mathrm{SG}(n)$, which we view as the group of permutations of the set $\{1, ..., n\}$ together with an *even* number of sign changes. The parity of the number of sign changes can be characterized by the constraint $\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n = 1$. In analogy to (6.1) we have

$$D_W K(s, t)$$

$$= \sum_{\substack{\varepsilon_1 = \pm 1, ..., \varepsilon_n = \pm 1 \\ \varepsilon_1 \cdots \varepsilon_n = 1}} \sum_{\sigma \in \mathfrak{S}_n} (\det \sigma) \prod_{j=1}^{n} K(s_j, \varepsilon_{\sigma(j)} t_{\sigma(j)})$$

$$= \sum_{\substack{\varepsilon_1 = \pm 1, ..., \varepsilon_n = \pm 1 \\ \varepsilon_1 \cdots \varepsilon_n = 1}} \begin{vmatrix} K(s_1, \varepsilon_1 t_1) & K(s_1, \varepsilon_2 t_2) & \cdots & K(s_1, \varepsilon_n t_n) \\ K(s_2, \varepsilon_1 t_1) & K(s_2, \varepsilon_2 t_2) & \cdots & K(s_2, \varepsilon_n t_n) \\ \vdots & \vdots & & \vdots \\ K(s_n, \varepsilon_1 t_1) & K(s_n, \varepsilon_2 t_2) & \cdots & K(s_n, \varepsilon_n, t_n) \end{vmatrix}. \qquad (6.3)$$

In the case in which $K(x, y) = e^{xy}$ for $x, y \in \mathbb{R}$, we may proceed in a manner similar to (6.2) to obtain

$$D_W K(s, t) = 2^{n-1} \left[\det(\cosh(s_j t_k)) + \det(\sinh(s_j t_k)) \right].$$

Then $D_W K(s, t)$ is positive for all $s_1 > \cdots > s_{n-1} > |s_n|$ and $t_1 > \cdots > t_{n-1} > |t_n|$.

We now exhibit a function which is totally positive with respect to all $\mathrm{U}(n)$ but not totally positive with respect to $\mathrm{SO}(4)$. This makes it clear that total positivity with respect to $\mathrm{U}(n)$ is not the same notion as total positivity for other compact Lie groups.

6.4. Examples. (1) Let $K(x, y) = (1 - xy)^{-1}$, $|xy| < 1$. It is known [7, 12] that K is totally positive with respect to $\mathrm{U}(n)$. We prove that K is totally positive with respect to $\mathrm{SO}(5)$ but not totally positive with respect to $\mathrm{SO}(4)$.

Recall that a fundamental Weyl chamber for SO(4) is $\{(s_1, s_2): s_1 > |s_2| > 0\}$. By means of a straightforward calculation using (6.3) we find that for $s = (s_1, s_2)$ and $t = (t_1, t_2)$,

$$D_{W'}K(s, t) = 2(1 + s_1 s_2 t_1 t_2) \begin{vmatrix} (1 - s_1^2 t_1^2)^{-1} & (1 - s_1^2 t_2^2)^{-1} \\ (1 - s_2^2 t_1^2)^{-1} & (1 - s_2^2 t_2^2)^{-1} \end{vmatrix}.$$

Therefore, K is totally positive with respect to SO(4) if and only if the function $L(x, y) = (1 - x^2 y^2)^{-1}$, $|xy| < 1$, is TP_2 (that is, totally positive with respect to U(2)).

To prove that L is not TP_2, we proceed as follows. Recall that a positive function L is TP_2 if and only if the function $f(x) = L(x, y_1)/L(x, y_2)$ is monotone increasing whenever $y_1 > y_2$ (this is simply an alternative way of stating that $L(x_1, y_1)/L(x_1, y_2) \geqslant L(x_2, y_1)/L(x_2, y_2)$ for $x_1 > x_2$ and $y_1 > y_2$). Here, we have $f(x) = (1 - x^2 y_1^2)^{-1} (1 - x^2 y_2^2)$, so that

$$[\log f(x)]' = \frac{2x(y_1^2 - y_2^2)}{(1 - x^2 y_1^2)(1 - x^2 y_2^2)}.$$

Therefore $(\log f(x))' < 0$, hence $f'(x) < 0$, for $x < 0$, and we conclude that K is not totally positive with respect to SO(4).

In the case of SO(5), a calculation using (6.1) yields

$$D_{W'}K(s, t) = 4 s_1 s_2 t_1 t_2 \begin{vmatrix} (1 - s_1^2 t_1^2)^{-1} & (1 - s_1^2 t_2^2)^{-1} \\ (1 - s_2^2 t_1^2)^{-1} & (1 - s_2^2 t_2^2)^{-1} \end{vmatrix}.$$

Since a fundamental Weyl chamber for SO(5) is $\{(s_1, s_2): s_1 > s_2 > 0\}$, proceeding as in the case of SO(4) we find that K is totally positive with respect to SO(5).

(2) The difference between total positivity with respect to SO(4) and SO(5) is only one instance of the difference between total positivity with respect to the even and odd order orthogonal groups. We shall prove that the function $K(x, y) = \sinh(xy)$ is not totally positive with respect to any SO($2n$). Starting with (6.3), and noting that $\sinh(\varepsilon x) = \varepsilon \sinh x$ for $\varepsilon = \pm 1$, we see that in the case of SO($2n$)

$$\begin{aligned} D_{W'}K(s, t) &= \sum_{\substack{\varepsilon_1 = \pm 1, \dots, \varepsilon_n = \pm 1 \\ \varepsilon_1 \cdots \varepsilon_n = 1}} \varepsilon_1 \cdots \varepsilon_n \det(\sinh(s_i t_j)) \\ &= 2^{n-1} \det(\sinh(s_i t_j)). \end{aligned}$$

It was proved earlier that the function K is totally positive with respect to SO($2n+1$); that is, this last determinant is positive whenever $s_1 > \cdots > s_n > 0$ and $t_1 > \cdots > t_n > 0$. Consequently the determinant is always negative if $s_1 > \cdots > s_{n-1} > 0 > s_n$ and $t_1 > \cdots > t_n > 0$. Therefore, $D_{W'}K(s, t)$ attains negative values on the fundamental region $s_1 > \cdots > s_{n-1} > |s_n|$ and $t_1 > \cdots > t_{n-1} > |t_n|$, and K is not totally positive with respect to $SO(2n)$.

At this stage, all our examples have been of functions which are totally positive with respect to Weyl groups of compact Lie groups. It remains to construct examples of functions which are totally positive with respect to irreducible finite reflection groups other than Weyl groups of compact Lie groups. From the classification theory described at the end of Section 2, we find that this problem asks for examples of functions that are totally positive with respect to finite reflection groups of type H_n^2 (with $n \geqslant 5$, $n \neq 6$), I_3 and I_4. For instance, we conjecture that the basic example treated in Proposition 6.1, $K(x, y) = e^{xy}$ for $(x, y) \in \mathbb{R}^2$, is strictly totally positive with respect to all finite reflection groups. If this conjecture is correct, one desires a proof that is valid for all irreducible finite reflection groups.

The case of root systems of type H_n^2 is particularly intriguing. In this case, the conjecture of total positivity of the function $K(x, y) = e^{xy}$, $x, y \in \mathbb{R}$, reduces to an unknown inequality for the classical Bessel functions.

6.5. *Root Systems of Type H_n^2.* For $n \geqslant 5$, $n \neq 6$, a root system of type H_n^2 is the set of vectors

$$\Delta = \left\{ \left(\cos \frac{j\pi}{n}, \sin \frac{j\pi}{n} \right) : j = 0, 1, ..., 2n-1 \right\}.$$

A system of positive roots is

$$\Delta_+ = \left\{ e_j = \left(\cos \frac{j\pi}{n}, \sin \frac{j\pi}{n} \right) : j = 0, 1, ..., n-1 \right\}$$

a base for Δ is

$$\Psi = \left\{ (1, 0), \left(\cos \frac{(n-1)\pi}{n}, \sin \frac{(n-1)\pi}{n} \right) \right\}$$

and a fundamental Weyl chamber for Δ is

$$\mathfrak{C} = \left\{ (s_1, s_2) \in \mathbb{R}^2 : s_1 > 0, s_1 \cos \frac{(n-1)\pi}{n} + s_2 \sin \frac{(n-1)\pi}{n} > 0 \right\}.$$

A simpler formulation is obtained by identifying $\mathbb{R}^2$ with $\mathbb{C}$ and using polar coordinates. Then we may identify Δ with the set of unit vectors, $\Delta = \{ e^{j\pi i/n} : j = 0, 1, ..., 2n-1 \}$, and make analogous identifications for Δ_+ and Ψ. The fundamental Weyl chamber is then identified with the set of vectors

$\mathfrak{C} = \{re^{i\theta}: r > 0,\ (n-2)\pi/2n < \theta < \pi/2\}$, and the reflection group W is realized as the dihedral group (with n reflections) acting simply transitively on the set of Weyl chambers.

Working in polar coordinates with $s = r_1 e^{i\theta_1}$ and $t = r_2 e^{i\theta_2}$, C. Dunkl pointed out to us that, after some elementary manipulations, the strict total positivity of the function $K(x, y) = e^{xy}$ is equivalent to the inequality

$$\sum_{m=0}^{\infty} \frac{(r_1 r_2)^{2m}}{m!\, 2^{2m}} \sum_{j=1}^{\infty} \frac{(r_1 r_2)^{jn}}{(m+jn)!\, 2^{jn}} \sin(jn\theta_1) \sin(jn\theta_2) > 0 \tag{6.4}$$

whenever $r_1, r_2 > 0$ and $0 < \theta_1, \theta_2 < \pi/n$. Setting $r = r_1 r_2/2$, replacing θ_j by θ_j/n, $j = 1, 2$, reversing the order of summation, and evaluating the sum over m, then (6.4) reduces to the inequality

$$\sum_{j=1}^{\infty} I_{jn}(r) \sin(j\theta_1) \sin(j\theta_2) > 0 \tag{6.5}$$

for $r > 0$, $\theta_1, \theta_2 < \pi$, where $I_n(r)$ is the Bessel function of imaginary argument. Applying a theorem of Fejer (cf. Askey [1, Theorem 1.2] or Gasper [4, (1.15)]), we deduce that (6.5) is equivalent to the inequality

$$\sum_{j=1}^{\infty} j I_{jn}(r) \sin j\theta > 0 \tag{6.6}$$

for $0 < \theta < \pi$, $r > 0$. It is unknown whether (6.6) is valid for general n, but Gasper has shown us how to prove it for $n = 1, 2$ using classical formulas for the Bessel functions. In any event (6.6) is valid for $n = 1, 2, 3, 4, 6$ since, for these values of n, the dihedral group is the Weyl group of a compact Lie group, so that Theorem 6.1 applies.

7. An FKG-Type Inequality for SO(5)

In the introduction, it was noted that the classical theory of total positivity plays a central role in the derivation of correlation inequalities on $\mathbb{R}^n$. Of fundamental importance in this area is the FKG inequality which is applied in quantum physics to the study of phase transitions; and in multivariate statistics for the construction of confidence intervals and other inferential tools. In this section, we prove an analogue of the classical FKG inequality using the concept of total positivity with respect to the Weyl group of SO(5).

We first recall the classical FKG correlation inequality on $\mathbb{R}^n$ (cf. [2, 10, 13]).

7.1. DEFINITION. (i) For $s=(s_1, ..., s_n)$ and $t=(t_1, ..., t_n)$ in $\mathbb{R}^n$, define the lattice operations $\vee$ and $\wedge$ by

$$s \vee t = (\max(s_1, t_1), ..., \max(s_n, t_n)), \qquad s \wedge t = (\min(s_1, t_1), ..., \min(s_n, t_n)).$$

(ii) A function $K: \mathbb{R}^n \to \mathbb{R}_+$ is *multivariate totally positive of order* 2 (MTP_2) if for all $s, t \in \mathbb{R}^n$,

$$K(s)\, K(t) \leqslant K(s \vee t)\, K(s \wedge t). \tag{7.1}$$

(iii) A function $\phi: \mathbb{R}^2 \to \mathbb{R}$ is *increasing* (decreasing) if ϕ is monotone increasing (decreasing) in each component.

Note that for $n=2$, (7.1) reduces to the classical definition of TP_2. Further, it is well-known that a differential condition that characterizes the *positive* MTP_2 functions is

$$\frac{\partial^2}{\partial s_i \partial s_j} \log K(s) \geqslant 0 \tag{7.2}$$

for all $i \neq j$ [2, 10, 13].

The following result is the statement of the classical FKG inequality.

7.2. THEOREM [2, 10, 13]. *Let K be a MTP_2 probability density function on $\mathbb{R}^n$. If $\phi_j: \mathbb{R}^2 \to \mathbb{R}$, $j=1, 2$, are both increasing or decreasing, then*

$$\int_{\mathbb{R}^2} \phi_1(s)\, \phi_2(s)\, K(s)\, ds \geqslant \left(\int_{\mathbb{R}^2} \phi_1(s)\, K(s)\, ds\right)\left(\int_{\mathbb{R}^2} \phi_2(s)\, K(s)\, ds\right) \tag{7.3}$$

whenever the integrals exist.

To develop an SO(5)-analog of the FKG inequality, we first need to define the concept of multivariate total positivity of order 2 with respect to SO(5). For $n \geqslant 2$, we denote by w_0 the reflection acting on $\mathbb{R}^n$ given by $w_0 \cdot (t_1, t_2, ..., t_n) = (-t_1, t_2, ..., t_n)$.

7.3. DEFINITION. For $n \geqslant 2$, a function $K: \mathbb{R}^n \to \mathbb{R}$ is MTP_2 *with respect to* SO(5) if

$$0 \leqslant \begin{vmatrix} K(s \vee t) & K(s) \\ K(t) & K(s \wedge t) \end{vmatrix} - \begin{vmatrix} K(w_0 \cdot (s \vee t)) & K(w_0 \cdot s) \\ K(t) & K(s \wedge t) \end{vmatrix}$$
$$- \begin{vmatrix} K(s \vee t) & K(s) \\ K(w_0 \cdot t) & K(w_0 \cdot (s \wedge t)) \end{vmatrix} - \begin{vmatrix} K(w_0 \cdot (s \vee t)) & K(w_0 \cdot s) \\ K(w_0 \cdot t) & K(w_0 \cdot (s \wedge t)) \end{vmatrix}. \tag{7.4}$$

Then the FKG inequality for SO(5) is as follows.

7.4. THEOREM. *Suppose that the function* $K: \mathbb{R}^n \to \mathbb{R}$ *is* MTP_2 *with respect to* $SO(5)$. *Assume that* $L(s) = K(s) - K(w_0 \cdot s)$ *is integrable (or without loss of generality, a probability density function) on the set* $\mathbb{S} = \{s \in \mathbb{R}^n: K(s) \geq K(w_0 \cdot s)\}$. *If* $\phi_j: \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2$, *are both increasing or decreasing on* $\mathbb{S}$ *then*

$$\int_{\mathbb{S}} \phi_1(s)\, \phi_2(s)\, L(s)\, ds \geq \left(\int_{\mathbb{S}} \phi_1(s)\, L(s)\, ds\right)\left(\int_{\mathbb{S}} \phi_2(s)\, L(s)\, ds\right) \qquad (7.5)$$

whenever the integrals converge.

Proof. On expanding the determinants in (7.4) and collecting terms, we find that (7.4) holds if and only if the function $L(s)$ satisfies the classical FKG condition (7.1); that is,

$$L(s)\, L(t) \leq L(s \vee t)\, L(s \wedge t)$$

for all $s, t \in \mathbb{R}^n$. Therefore, (7.5) follows from Theorem 7.2.

7.5. EXAMPLE. Let $n = 2$, $0 < \rho \leq 1$ and

$$K(x, y) = \tfrac{1}{2} \exp(-\tfrac{1}{2}(x^2 + y^2) + \rho xy)$$

for $(x, y) \in \mathbb{R}^2$. The function K is a constant multiple of the bivariate normal probability density function with mean $(0, 0)$ and covariance matrix

$$\Sigma = \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}.$$

Then

$$L(x, y) = \exp(-\tfrac{1}{2}(x^2 + y^2)) \sinh(\rho xy)$$

so that $\mathbb{S} = \{(x, y): xy > 0\}$. Clearly, L is integrable on $\mathbb{S}$. After normalization of L, the FKG inequality (7.5) holds once it has been proved that L, equivalently the function $\sinh(\rho xy)$, is MTP_2 on $\mathbb{S}$. Without loss of generality, we take $\rho = 1$. To determine the subset of $\mathbb{S}$ on which the function $\sinh(xy)$ is MTP_2, we apply the differential condition (7.2). Since

$$\sinh^2(xy) \frac{\partial^2}{\partial x\, \partial y} \log \sinh(xy) = -\frac{1}{2}(2xy - \sinh(2xy)) \geq 0$$

for $xy > 0$, then it follows that L is MTP_2 on all of $\mathbb{S}$.

We can also obtain an analog of the Holley–Preston–Kemperman inequality [13, Theorem 2.2].

7.6. THEOREM. *Let K_1, K_2 be functions on $\mathbb{R}^n$, $n \geqslant 2$, such that*

$$0 \leqslant \begin{vmatrix} K_2(s \vee t) & K_1(s) \\ K_2(t) & K_1(s \wedge t) \end{vmatrix} - \begin{vmatrix} K_2(w_0 \cdot (s \vee t)) & K_1(w_0 \cdot s) \\ K_2(t) & K_1(s \wedge t) \end{vmatrix}$$
$$- \begin{vmatrix} K_2(s \vee t) & K_1(s) \\ K_2(w_0 \cdot t) & K_1(w_0 \cdot (s \wedge t)) \end{vmatrix} + \begin{vmatrix} K_2(w_0 \cdot (s \vee t)) & K_1(w_0 \cdot s) \\ K_2(w_0 \cdot t) & K_1(w_0 \cdot (s \wedge t)) \end{vmatrix}. \tag{7.6}$$

Let $L_j(s) = K_j(s) - K_j(w_0 \cdot s)$, $j = 1, 2$, $\mathbb{S} = \{s \in \mathbb{R}^n : L_j(s) > 0, j = 1, 2\}$, and assume that L_1 and L_2 are probability density functions on $\mathbb{S}$. Then for any increasing function ϕ on $\mathbb{S}$,

$$\int_{\mathbb{S}} \phi(s)\, L_1(s)\, ds \leqslant \int_{\mathbb{S}} \phi(s)\, L_2(s)\, ds.$$

Similar to the proof of Theorem 7.4, Theorem 7.6 is proved by noting that (7.6) is equivalent to

$$L_2(s \vee t)\, L_1(s \wedge t) - L_2(t)\, L_1(s) \geqslant 0$$

for $s, t \in \mathbb{R}^n$, and then applying [13, Theorem 2.2].

ACKNOWLEDGMENTS

The authors are grateful to Charles Dunkl for helpful conversations on the theory of finite reflection groups, to Eric Opdam for discussions on the subject of Harish-Chandra's integral formula, and to George Gasper for comments on Example 6.5.

REFERENCES

1. R. ASKEY, Orthogonal polynomials and special functions, *in* "Regional Conference Series in Applied Math.," Vol. 21, SIAM, Philadelphia, PA, 1975.
2. G. A. BATTLE AND L. ROSEN, The FKG inequality for the Yukawa quantum field theory, *J. Statist. Phys.* **22** (1980), 123–192.
3. T. BRÖCKER AND T. TOM DIECK, "Representations of Compact Lie Groups," Springer, New York, 1985.
4. G. GASPER, Banach algebras for Jacobi series and positivity of a kernel, *Ann. Math.* **95** (1972), 261–280.
5. L. C. GROVE AND C. T. BENSON, "Finite Reflection Groups," 2nd ed., Springer, New York, 1985.

6. K. I. GROSS AND D. ST. P. RICHARDS, Special functions of matrix argument. I. Algebraic induction, zonal polynomials and hypergeometric functions, *Trans. Amer. Math. Soc.* **301** (1987), 781–811.
7. K. I. GROSS AND D. ST. P. RICHARDS, Total positivity, spherical series, and hypergeometric functions of matrix argument, *J. Approx. Theory* **59** (1989), 224–246.
8. HARISH-CHANDRA, Differential operators on a semisimple Lie algebra, *Amer. J. Math.* **79** (1957), 87–120; also *in* "Collected Works," Vol. 2 (V. S. Varadarajan, Ed.), Springer, New York.
9. S. HELGASON, "Groups and Geometric Analysis," Academic Press, New York, 1984.
10. I. HERBST AND L. PITT, Diffusion equation techniques in stochastic monotonicity and positive correlations, *Probab. Theory Related Fields* **87** (1991), 275–312.
11. J. E. HUMPHREYS, "Introduction to Lie Algebras and Representation Theory," Springer, New York, 1972.
12. S. KARLIN, "Total Positivity," Vol. 1, Stanford Univ. Press, Stanford, CA.
13. S. KARLIN AND Y. RINOTT, Classes of orderings of measures and related correlation inequalities. I. Multivariate totally positive distributions. *J. Multivariate Anal.* **10** (1990), 467–498.
14. S. KARLIN AND Y. RINOTT, A generalized Cauchy-Binet formula and applications to total positivity and majorization, *J. Multivariate Anal.* **27** (1988), 284–299.
15. A. W. KNAPP, "Representation Theory of Semisimple Groups: An Overview Based on Examples," Princeton Univ. Press, Princeton, NJ, 1986.
16. D. ST. P. RICHARDS, Totally positive kernels, Pólya frequency functions, and generalized hypergeometric series, *Linear Algebra and Appl.* **137/138** (1990), 467–478.
17. J. R. STEMBRIDGE, Immanants of totally positive matrices are nonnegative, *Bull. London Math. Soc.* **23** (1991), 422–428.
18. N. WALLACH, "Harmonic Analysis on Homogeneous Spaces," Dekker, New York, 1973.